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Evolution equations for counterpropagating edge waves

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Asymptotically exact evolution equations for counterpropagating shallow-water edge waves are derived. The structure of the equations depends only on the symmetries of the problem and on the fact that the group velocity of the edge waves is of order one. As a result the equations take the form of parametrically forced Davey–Stewartson equations with mean-field coupling. The calculations extend existing work on parametric excitation of edge waves by normally incident waves to arbitrary beach profiles with asymptotically constant depth, and include coupling to wave-generated mean longshore currents. Dissipation arises generically from radiation damping, but we also consider heuristically the effects of linear boundary-layer damping. Spatially modulated waves do not couple to the parametric forcing due to the non-locality of the evolution equations and are damped. Thus only spatially uniform wavetrains are expected as stable solutions. If linear dissipation is included the parametric coupling selects standing waves, but in the undamped case travelling wave states are possible. Both classes of solutions are examined for modulational instabilities, and stability conditions for the generic evolution equations are presented. However, modulational instability is found to be excluded in the shallow-water formulation through the effects of the mean flow. Explicit numerical results for two experimentally relevant beach profiles, exponentially decaying and piecewise linear, are presented.

1. Introduction

This paper is a contribution to the study of the stability and dynamics of edge waves driven parametrically by onshore waves. The first discussion of linear, coastally trapped waves dates back to Stokes in 1846. Much of the subsequent analysis makes use of simplifying assumptions designed to deal with several of the harder or more technical aspects of the analysis, including general beach profiles and the transition to deep water, radiation and viscous damping, wave-driven mean flows, and the dynamics of the interaction between counterpropagating edge waves. Certainly much of the work is self-consistent within the framework postulated; for instance, Minzoni & Whitham (1977) carried through their analysis without the shallow-water assumption, but considered only undamped standing waves on a beach of constant slope. Recently a number of papers have addressed some of the other problems. Miles (1990*b*) and Mathew & Akylas (1990) have partially accounted for radiation and viscous damping, general profiles and the transition to deep water, while Akylas (1983) and Miles (1991) have addressed the dynamics of counterpropagating Stokes edge waves. The present

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paper is closest in spirit to that of Akylas (1983) in that we focus on deriving envelope equations for parametrically driven counterpropagating edge waves. Akylas uses the full water wave equations, but is only able to treat beach profiles of constant slope $\pi/2N$, where N is an integer, and does not consider wave-driven mean flows. On the other hand, the shallow-water formulation that we use enables us to consider arbitrary beach profiles, provided that these asymptote to a constant depth in the offshore direction. Minzoni (1976) has shown that in this case the shallow-water formulation preserves the essential characteristics of the deep-water theory. More importantly, we show that the equations derived by Akylas (1983) require modification because the group velocity is $O(1)$. As a result the asymptotically correct amplitude equations contain non-local terms which modify the stability properties of non-uniform trains of standing edge waves. In addition we emphasize the importance of wave-driven mean flows in the edge wave problem, an effect largely ignored in the literature, except as discussed by Miles (1991). The equations we derive for the amplitudes of the two counterpropagating trains of edge waves include self-consistently the coupling to such flows. Finally, we relate the problem of parametric excitation of edge waves to existing work on the Faraday system, emphasizing in particular that the parametric forcing favours standing edge waves over travelling edge waves, at least for normal incidence of the external wave field and moderate detuning of the forcing frequency.

Throughout this paper we assume that the beach profile and incident wave field are uniform in the longshore direction. The basic structure of the amplitude equations describing the interaction of counterpropagating edge waves then follows from the observation that, with periodic boundary conditions in the longshore direction and normal incidence of the external wave field, the system has O(2) symmetry. This is the symmetry of a circle under rotations and reflections, the rotations being identified with translations in the longshore direction modulo the spatial period, and the reflections with those in vertical planes in the offshore direction. As a result of the O(2) symmetry in the longshore direction, the competition between counterpropagating waves in the longshore direction selects either a standing wavetrain (which maintains the reflection symmetry) or a travelling wavetrain (with the upshore and downshore wavetrains related by reflection symmetry). This wave selection process is nonlinear. In the absence of forcing, however, such waves decay because of both viscous and radiation damping. To maintain the wave against decay, external forcing by an incident wavetrain is necessary. The excitation mechanism is parametric in nature, with edge waves of frequency ω and amplitude $O(\epsilon)$ being excited by incident waves of frequency near 2ω and amplitude ϵ^2. With normal incidence, this parametric forcing respects the O(2) symmetry and results in a problem of the type studied by Riecke, Crawford & Knobloch (1988). For spatially uniform wavetrains these authors showed that the parametric forcing couples the two counterpropagating waves and as a result favours the existence of standing waves. In the present paper we confirm this result in the context of edge waves and show explicitly how our evolution equations reduce to the required O(2) equivariant form for uniform wavetrains. However, the stability of non-uniform trains of standing waves in dispersive systems is complicated by the presence of a non-zero group velocity. When the group velocity is $O(\epsilon)$ the resulting problem is described (in the Hamiltonian case) by a pair of nonlinear Schrödinger equations with local coupling (cf. Akylas 1983). This is no longer the case when the group velocity is $O(1)$, with the coupling now being of mean-field type (cf. Knobloch & Gibbon 1991; Knobloch, 1992; Pierce & Knobloch 1993).

The paper is organized as follows. In §2 we derive the asymptotically exact evolution equations for counterpropagating shallow-water edge waves. In §3 we generalize these equations slightly to allow for viscous dissipation and small detuning of wavenumber and frequency. We than analyse the existence and stability of both spatially homogeneous and inhomogeneous wavetrains. There are significant differences between the results of our analysis for spatially modulated solutions and previous work. This is due in part to the non-locality of the evolution equations, and in part to the strong coupling between the edge waves and the mean flow in the bulk. Explicit computations for two beach profiles are performed in §4. In the final section we argue that spatially inhomogeneous edge waves require a compatible mean flow in the bulk which can support the modulations at the beach, and that, within the shallow-water equations, it is apparently impossible to generate such a mean flow. We conclude that the coupling of edge waves with longshore flows favours spatially homogeneous wavetrains.

2. Asymptotic expansion

Throughout this paper we assume that the inviscid shallow-water equations are uniformly valid in the domain $\{0 \leqslant x < \infty, -\infty < y < \infty\}$, where (x, y) are the offshore and longshore coordinates, respectively. The equations are

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = 0, \quad \zeta_t + \nabla\cdot[(h+\zeta)\nabla\phi] = 0, \tag{1 a, b}$$

where $h \geqslant 0$ is the depth (measured downwards), ζ is the surface elevation, and ϕ is the horizontal velocity potential. The operator ∇ denotes $(\partial/\partial x, \partial/\partial y)$. Eliminating ζ one obtains an equation for ϕ alone:

$$g\nabla\cdot(h\nabla\phi) - \phi_{tt} = \nabla\phi\cdot\nabla\phi_t + \nabla\cdot[(\phi_t + \tfrac{1}{2}|\nabla\phi|^2)\nabla\phi]. \tag{2}$$

We assume that the beach profile, $h(x)$, is such that $h(0) = 0, h'(0) \neq 0$, and that it asymptotes monotonically to a constant value, h_0, beyond a fixed $O(1)$ distance from the shore, x_0. The boundary conditions on the potential require it to be regular at $x = 0$, allowing for oscillatory run-up but not breaking, and bounded in the far field as $x \to \infty$. We choose dimensions such that $h_0 = g = 1$, where g is the acceleration due to gravity. We now introduce the slow variables

$$Y = \epsilon y, \quad T = \epsilon t, \quad \tau = \epsilon^2 t, \tag{3 a}$$

and expand ϕ and ζ in an asymptotic and harmonic series:

$$\phi = \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n}\sum_{l=-n}^{l=n} \epsilon^n \phi_{[nml]}\, \mathrm{e}^{\mathrm{i}mky}\, \mathrm{e}^{-\mathrm{i}l\omega t}, \tag{3 b}$$

$$\zeta = \sum_{n=1}^{\infty}\sum_{m=-n}^{m=n}\sum_{l=-n}^{l=n} \epsilon^n \zeta_{[nml]}\, \mathrm{e}^{\mathrm{i}mky}\, \mathrm{e}^{-\mathrm{i}l\omega t}, \tag{3 c}$$

where

$$\phi_{[nml]} \equiv \phi_{[nml]}(x, Y, T, \tau) = \bar{\phi}_{[n(-m)(-l)]}, \tag{3 d}$$

$$\zeta_{[nml]} \equiv \zeta_{[nml]}(x, Y, T, \tau) = \bar{\zeta}_{[n(-m)(-l)]}. \tag{3 e}$$

Here k is the wavenumber of the edge waves and ω their frequency; in this formulation the frequency of the incident waves is 2ω. Note that although it is possible to introduce the slow variable ϵx as well, we have found that this is unnecessary and results in no greater generality. This is a consequence of the uniform validity of the shallow-water equations in the present problem and the assumption that the incident wavetrain and the depth profile are unmodulated on scales of $O(\epsilon^{-1})$. Substituting the expansion (3*b*) into the partial differential equation (2) and collecting powers of ϵ, e^{iky} and $e^{-i\omega t}$, one finds that the various terms must have the following forms:

$$\phi_{[11(\pm 1)]} = A^{\pm}(Y,T,\tau)\,a, \quad \phi_{[100]} = B(Y,T,\tau)\,b, \quad \phi_{[21(\pm 1)]} = D^{\pm}(Y,T)\,a + \mathrm{i}A^{\pm}_{Y}\,p, \tag{4a–c}$$

$$\phi_{[220]} = A^{+}A^{-}s, \quad \phi_{[202]} = j + \mathrm{i}A^{+}\overline{A^{-}}q, \quad \phi_{[22(\pm 2)]} = \pm\mathrm{i}A^{\pm 2}r, \tag{4d–f}$$

$$\phi_{[200]} = C(Y,T)\,c, \quad \phi_{[300]} = d, \quad \phi_{[31(\pm 1)]} = e^{\pm}. \tag{4g–i}$$

Here an upper case letter denotes an amplitude which is a function of the slow variables as indicated, while a lower case denotes a function of the fast offshore coordinate, x. No other amplitudes enter in the subsequent calculations. The amplitudes $A^{\pm}$ describe the slow evolution of upshore and downshore travelling edge waves and are the quantities of primary interest in the present paper. These amplitudes will be found to couple to the leading-order mean flow whose longshore variation is specified by the quantity B. Consequently, in the following we seek dynamical equations linking the evolution of the three amplitudes $A^{\pm}$ and B. This derivation will be carried out for a coastally trapped wave whose decay in the offshore direction is specified by $a(x)$. This function may, without loss of generality, be taken to be real, and is an eigenfunction of a boundary value problem given below. The functions $q(x)$ and $j(x)$ represent radiated waves and a superposition of incoming and reflected waves propagating normally to the beach, respectively. Thus j represents the forcing of the edge waves by an incident wave. In the absence of viscous dissipation, j is sinusoidal in the far field. Note that this wave is uniform in y and has amplitude $O(\epsilon^2)$ and frequency 2ω. We have allowed a slight loss of generality in not providing a slowly varying amplitude for j. Foda & Mei (1988) consider the effects of such variations on scales of $O(\epsilon^{-1})$, but we observe that there is no self-consistent mechanism for generating these variations away from the boundary when the incident wave field has a steepness of $O(\epsilon^2)$.

For each order and harmonic, we have a boundary value problem for the fast offshore dependence on the semi-infinite interval $x \in [0, +\infty]$. At order $[11(\pm 1)]$ we obtain the linear problem

$$(\omega^2 - k^2 h)\,a + (ha')' = 0. \tag{5}$$

This equation is a Sturm–Liouville eigenvalue problem for ω (alternatively, k). Since $h'(0) \neq 0$, the equation has regular singular points at $x = \{0, +\infty\}$, with zero exponents at $x = 0$. We impose the requirement that the edge-wave eigenfunction, a, be in L^2 (coastal trapping) and regular at $x = 0$ (no wave breaking), thereby restricting ω to the point spectrum $0 < \omega^2 < k^2$ (Minzoni 1976). Consequently $a(\infty) = 0$, and we choose the normalization such that $a(0) = 1$. The point spectrum consists of a finite set of real positive eigenvalues, $\omega(k)$, whose precise number depends on the depth profile and the wavenumber. This is very different from 'shallow water' formulations using equations (1), but where the depth is allowed to increase to infinity with the distance from shore; as discussed by Minzoni (1976) these formulations are anomalous, allowing a countable infinity of modes for all k. Specifying the frequency of the incident wave and the mode number of the linear solution automatically selects the wavenumber of the edge wave. Conversely, selecting the longshore wavenumber forces a choice of frequency, though weak detuning is allowed via the multiple-scales expansion. Typically, in some range of k the selected mode will be unique. When more than one mode is excited by the incident wave, the addition of viscous damping will shift the eigenvalues off the real axis by different amounts, thereby selecting the preferred mode number (cf. Guza & Davis 1974). When k is assumed fixed and ω is increased, the mode that is selected in this way is the first one. For a monotonic beach profile, this mode has no zeros, while the second mode has one zero, etc., a result that follows from the oscillation theorem for the Sturm–Liouville problem (5) (Weinberger 1965). Thus the first mode corresponds to the simplest, intuitive flow pattern while higher modes represent 'cellular' patterns in the offshore direction. When ω is held fixed, however, the mode selection process is more involved, and no general statements can apparently be made.

At order $[21(\pm 1)]$ one finds that $A^{\pm}_{T} \pm c_g A^{\pm}_{Y} = 0$, with the constant c_g identified with the edge-wave group velocity and determined from the solvability condition for the equation

$$(\omega^2 - k^2 h)p + (hp')' = 2(c_g \omega - kh)\, a. \tag{6a}$$

This equation will have a solution in L^2 if and only if

$$c_g = \frac{k}{\omega}\int_0^\infty ha^2\,\mathrm{d}x \bigg/ \int_0^\infty a^2\,\mathrm{d}x. \tag{6b}$$

The corresponding solution for p may be formally obtained as a sum over the remaining discrete eigenvalues plus an integral over the continuous spectrum. In contrast to the q-equation discussed below, there will be no radiation generated by this integral, since the inhomogeneous term is in L^2 and the eigenvalue of (5) does not intersect the continuous spectrum. The requirement $A^{\pm}_{T} \pm c_g A^{\pm}_{Y} = 0$ is implicit in the work of Akylas (1983) and was also noted by Foda & Mei (1988). Mathew & Akylas (1990) and Miles (1991) consider only edge waves that are spatially uniform so that this requirement does not arise in their expansions.

The remaining boundary value problems up to $O(\epsilon^2)$ are

$$(hb') = 0, \quad (hc') = 0, \tag{7a, b}$$

$$4\omega^2 j + (hj')' = 0, \quad 4\omega^2 q + (hq')' = -2\omega(k^2a^2 + 2a'^2 + aa''), \tag{7c, d}$$

$$-4k^2 hs + (hs')' = 0, \quad 4(\omega^2 - k^2 h)r + (hr')' = \omega(3k^2a^2 - 2a'^2 - aa''). \tag{7e, f}$$

A priori, the only boundary conditions placed on the solutions are that they be bounded at $x = 0$ and $x = \infty$. More precise conditions, appropriate to evanescent and radiating solutions, will be discussed and applied below. The bounded solutions of (7a, b) are $b(x) = c(x) \equiv 1$. The j- and q-equations have no point spectrum. Thus q may be formally constructed as an integral over the continuous spectrum, while j is a homogeneous solution which must satisfy the boundary conditions on the external wave field at $x = \infty$. However, the eigenvalue of (5) will always intersect the continuous spectrum of the linear operator in (7d), so q will have a simple pole in its eigenfunction transform. This pole will generate a solution that is not evanescent and corresponds to radiation. The integration contour must therefore be chosen appropriately to satisfy the radiation boundary condition. This integral cannot be evaluated in closed form for most profiles. Miles (1990b) obtains an approximate solution by truncating the expansion, while we avoid the issue by obtaining numerical solutions (see §4). It is important to note that as long as the eigenfunction transform of the inhomogeneous terms does not have a zero at ω^2, this type of radiation will always be present.

The equation for s has no oscillatory solutions as $x \rightarrow \infty$. Multiplying (7e) by the complex conjugate of s and integrating by parts gives $s \equiv 0$. We note that this is a direct consequence of the fact that the two counterpropagating edge waves have identical offshore spatial dependence, since otherwise (7*e*) would have a non-vanishing inhomogeneous term. We will discuss this point further in §5.

The last equation, for r, will in general have both a discrete and a continuous spectrum. However, rescaling x by a factor of 2 gives $(\hat{\omega}^2-k^2h(\frac{1}{2}\hat{x}))\hat{r}+(h(\frac{1}{2}\hat{x})\hat{r}')'=0$ as the homogeneous part. The assumption that h is monotonic implies that $h(\frac{1}{2}x)<h(x)$, so that the monotonicity theorem (Weinberger 1965) gives the result that $\hat{\omega}_j<\omega_j$ for each eigenvalue and a given k. Thus there is no solvability criterion, and r may again be found by an eigenfunction transform. Since the continuous spectra of (5) and (7*f*) are the same, ω^2 will not intersect the continuous spectrum of (7*f*). Consequently, the transform of r will be analytic and there will be no radiation generated by this term. If the eigenvalue of (5) happens to intersect the point spectrum of (7*f*) because of a higher mode or a non-monotonic profile, we have a modal resonance problem, and the asymptotic expansion, (3), must be modified accordingly (McGoldrick 1970).

At third order in ϵ, we obtain three equations of importance. At [300] we have

$$(hd')'=B_{TT}-hB_{YY}+(c_g k^2a^2+2k\omega a^2+2c_g a'^2+c_g aa''-\omega(pa''-ap''))(|A^-|^2-|A^+|^2)_Y, \tag{8a}$$

while at $[31(\pm 1)]$:

$$\begin{aligned}(\omega^2-k^2h)e^++(he^{+\prime})'=&-2\mathrm{i}\omega aD_T^+-2\mathrm{i}kahD_Y^+-2\mathrm{i}\omega aA_\tau^++(c_g^2a-ah-2c_g\omega p+2khp)A_{YY}^+\\&+\mathrm{i}\omega A^-(2k^2aj-2a'j'-2ja''+aj'')+A^+|A^+|^2[\omega(6k^2ar+2a'r'+2ra''-ar'')\\&+\tfrac{1}{2}(-3k^4a^3+k^2aa'^2+k^2a^2a''+9a'^2a'')]+A^+|A^-|^2[\omega(-2k^2aq+2a'q'-2\mathrm{i}a's'\\&+2qa''-aq''-\mathrm{i}as'')+(-3k^4a^3+k^2aa'^2+k^2a^2a''+9a'^2a'')]+A^+[(-k^2a+a'')B_T\\&+2k\omega aB_Y],\end{aligned} \tag{8b}$$

and

$$\begin{aligned}(\omega^2-k^2h)e^-+(he^{-\prime})'=&\,2\mathrm{i}\omega aD_T^--2\mathrm{i}kahD_Y^-+2\mathrm{i}\omega aA_\tau^-+(c_g^2a-ah-2c_g\omega p+2khp)A_{YY}^-\\&+\mathrm{i}\omega A^+(-2k^2a\bar{j}+2a'\bar{j}'+2\bar{j}a''-a\bar{j}'')+A^-|A^-|^2[\omega(6k^2ar+2a'r'+2ra''-ar'')\\&+\tfrac{1}{2}(-3k^4a^3+k^2aa'^2+k^2a^2a''+9a'^2a'')]+A^-|A^+|^2[\omega(-2k^2a\bar{q}+2a'\bar{q}'+2\mathrm{i}a's'\\&+2\bar{q}a''-a\bar{q}''+\mathrm{i}as'')+(-3k^4a^3+k^2aa'^2+k^2a^2a''+9a'^2a'')]+A^-[(-k^2a+a'')B_T\\&-2k\omega aB_Y].\end{aligned} \tag{8c}$$

The solvability criteria require that the right-hand side of the $e^\pm$ equations (8*b*, *c*) be orthogonal to a in L^2, and that the integral of the right-hand side of the mean flow equation (8*a*) over $[0,\infty]$ must vanish. The latter gives two separate conditions:

$$B_{TT}-B_{YY}=0,\quad B_{YY}=\rho(|A^+|^2-|A^-|^2)_Y. \tag{9a, b}$$

Equation (9*a*) implies that any inhomogeneous or unsteady mean flow must take the form of a shallow-water wave with $O(\epsilon^{-1})$ wavelength and period. Equation (9*b*) implies that standing waves ($|A^+|=|A^-|$) are not associated with mean flows, but all other waves do generate such flows. The coefficient ρ is given below. The solvability conditions for (8*b*, *c*) yield evolution equations for $D^\pm$, the second-order corrections to the edge waves. In terms of the new longshore coordinates $\chi^\pm=Y\mp c_gT$ these take the form

$$\begin{aligned}-2\mathrm{i}c_g\nu D_{\chi^\mp}^\pm\mp\mathrm{i}\nu A_\tau^\pm+(\alpha_r\pm\mathrm{i}\alpha_i)A^\pm+(\beta_r\pm\mathrm{i}\beta_i)A^\mp+(\gamma_r\pm\mathrm{i}\gamma_i)A_{\chi^\pm\chi^\pm}^\pm+(\delta_r\pm\mathrm{i}\delta_i)A^\pm|A^\pm|^2\\+(\lambda_r\pm\mathrm{i}\lambda_i)A^\pm|A^\mp|^2+A^\pm[\theta_1B_T\pm k\nu B_Y]=0,\end{aligned} \tag{10a, b}$$

where $A^{\pm}_{\chi^{\mp}} = 0$. In (10a, b) the coefficients are given by integrals over the interval $[0, \infty]$:

$$\rho = \frac{kv - c_g \theta_1}{\int_0^\infty (1-h)\,\mathrm{d}x}, \tag{11a}$$

$$\alpha = 0, \tag{11b}$$

$$\beta = \mathrm{i}\omega \int_0^\infty a(2k^2 aj - 2a'j' - 2ja'' + aj'')\,\mathrm{d}x, \tag{11c}$$

$$\gamma = \int_0^\infty a(c_g^2 a - ah - 2c_g \omega p + 2khp)\,\mathrm{d}x, \tag{11d}$$

$$\delta = \tfrac{1}{2}\theta_2 + \omega \int_0^\infty a(6k^2 ar + 2a'r' + 2ra'' - ar'')\,\mathrm{d}x, \tag{11e}$$

$$\lambda = \theta_2 + \omega \int_0^\infty a(-2k^2 aq + 2a'q' + 2qa'' - aq'')\,\mathrm{d}x, \tag{11f}$$

$$\theta_1 = \int_0^\infty a(a'' - k^2 a)\,\mathrm{d}x, \tag{11g}$$

$$\theta_2 = \int_0^\infty a(-3k^4 a^3 + k^2 aa'^2 + k^2 a^2 a'' + 9a'^2 a'')\,\mathrm{d}x, \tag{11h}$$

$$v = 2\omega \int_0^\infty a^2\,\mathrm{d}x. \tag{11i}$$

The aa'', ap'' and pa'' terms in the expression for ρ have been reduced by an integration by parts, followed by a power series expansion of the regular solutions for a and p near $x = 0$ to obtain $p'(0)$ and $a'(0)$ in terms of $a(0)$ and $p(0)$. The final stage in the derivation is the observation that the expansion for $\phi(x, y, t)$ remains *asymptotic* for y and t of $O(\epsilon^{-2})$ if and only if the $D^{\pm}$ do not contain any secular terms in $\chi^{\pm}$. If this requirement is not met the ordering assumed in the expansion (3b) will be violated at y and t of $O(\epsilon^{-2})$. The resulting solvability conditions for $D^{\pm}$ take the form (Knobloch & Gibbon 1991; Knobloch 1992):

$$\mathrm{i}vA^+_\tau = \alpha A^+ + \beta \langle A^- \rangle_{\chi^-} + \gamma A^+_{\chi^+\chi^+} + \delta A^+ |A^+|^2 + \lambda A^+ \langle |A^-|^2 \rangle_{\chi^-} + A^+ \langle \theta_1 B_T + kvB_Y \rangle_{\chi^-}, \tag{12a}$$

$$-\mathrm{i}vA^-_\tau = \bar{\alpha} A^- + \bar{\beta} \langle A^+ \rangle_{\chi^+} + \bar{\gamma} A^-_{\chi^-\chi^-} + \bar{\delta} A^- |A^-|^2 + \bar{\lambda} A^- \langle |A^+|^2 \rangle_{\chi^+} + A^- \langle \theta_1 B_T - kvB_Y \rangle_{\chi^+}. \tag{12b}$$

Here the notation $\langle\ \rangle_{\chi^\pm}$ denotes an average over the variable $\chi^{\pm}$, either over a period in $\chi^{\pm}$ if periodic boundary conditions are imposed, or over the real line if not.

In order to proceed we need to evaluate the coupling to the mean flow, B. This flow appears linearly in both (9a, b) and (12a, b), and so may be written as a sum of the second-order Eulerian drifts associated with each wave plus an apparently arbitrary gauge which is linear function of T:

$$B \equiv B^+(\chi^+, \tau) + B^-(\chi^-, \tau) + \kappa(\tau)\,T. \tag{13a}$$

Integrating (9b) once with respect to Y gives

$$B^{+}_{\chi^+} = \rho |A^+|^2 \quad \text{and} \quad B^{-}_{\chi^-} = -\rho |A^-|^2. \tag{13b, c}$$

No constants of integration are added since $B^{\pm}$ are only generated in response to $A^{\pm}$, respectively. The addition of the linear function of T to B does not affect the physical velocity, B_Y, nor does it affect the shallow-water wave equation, (9a). We need this freedom, however, to make sure that the mean water level at $x = \infty$ remains unchanged by the presence of the edge waves. From (1a) we find that $\zeta_{[100]} = 0$, but that

$$\zeta_{[200]} = -B_T - (k^2 a^2 + a'^2)(|A^+|^2 + |A^-|^2). \tag{14}$$

This expression is commonly referred to as the 'set up' or 'set down' associated with the edge waves and is independent of any set up or 'storm surge' associated with the incident wavetrain. The second term represents an $O(\epsilon^2)$ change in the water volume per unit length of beach that has been observed experimentally (Yeh 1986). This volume change does not violate the conservation of mass, since it may be offset with an infinitesimal change of water level in the bulk. As $x \to \infty$ the second term falls off exponentially, however, and $\zeta_{[200]} \to -B_T$. Hence the requirement that the mean water level at $x = \infty$ remains unaffected is equivalent to the requirement that the spatial average of this term must be zero, $\langle B_T \rangle_Y = 0$. This in turn forces a specific choice of κ:

$$\kappa = \rho c_g (\langle |A^+|^2 \rangle_{\chi^+} + \langle |A^-|^2 \rangle_{\chi^-}). \tag{15}$$

Using (13) and (15) the derivatives of B may be written in terms of $A^{\pm}$, and B eliminated from (12a, b):

$$\mathrm{i}\nu A^+_\tau = \alpha A^+ + \beta \langle A^- \rangle_{\chi^-} + \gamma A^+_{\chi^+\chi^+} + \Delta A^+ |A^+|^2 + \Theta A^+ \langle |A^+|^2 \rangle_{\chi^+} + \Lambda A^+ \langle |A^-|^2 \rangle_{\chi^-}, \tag{16a}$$

$$-\mathrm{i}\nu A^-_\tau = \bar{\alpha} A^- + \bar{\beta} \langle A^+ \rangle_{\chi^+} + \bar{\gamma} A^-_{\chi^-\chi^-} + \bar{\Delta} A^- |A^-|^2 + \Theta A^- \langle |A^-|^2 \rangle_{\chi^-} + \bar{\Lambda} A^- \langle |A^+|^2 \rangle_{\chi^+}, \tag{16b}$$

where $\Delta \equiv \delta + \rho(k\nu - c_g \theta_1)$, $\Theta \equiv \rho c_g \theta_1$, and $\Lambda \equiv \lambda - \rho k \nu$. Equations (16$a$, b), together with (9a, b) and (13), constitute the desired amplitude equations for counter-propagating edge waves, and are a pair of non-local equations with mean-field coupling, and an additional coupling to the mean flow. In the absence of parametric forcing ($\beta = 0$), similar equations have recently been derived by Matkowsky & Volpert (1992) in their study of instabilities of a propagating combustion front. Generically, all the coefficients in (16) will be complex, except for Θ and ν. However, from (7) we see that only q and j are potentially complex, since they must satisfy non-trivial boundary conditions at $x = \infty$. Thus within the model system defined by (1) the coefficients γ and Δ are also real, and only the coefficient Λ is complex. Moreover, the coefficient β may be made real by rotating the phases of $A^{\pm}$, and we assume hereafter that this has been done. Nonetheless, we have written the equations in the general form (16), anticipating that in more general circumstances the coefficients α, γ and Δ will also be complex (cf. Miles & Henderson 1990). Note in particular that, as derived, the evolution equations have $\alpha = 0$. This is a consequence of assuming an exact 2:1 resonance between the incident wave and the edge wave. More generally an incident wave of frequency 2ω excites edge waves of frequency $\omega - \epsilon^2 \alpha_r / \nu$, where α_r is a detuning parameter. Such a detuning may arise from a variety of mechanisms, such as viscosity or external constraints on the wavenumber of the fundamental edge wave. More fundamentally, however, it describes the effect of longshore modulation on $O(\epsilon^{-2})$ lengthscales. This can be seen by introducing a slow lengthscale $Y' \equiv \epsilon^2 y$ (cf. Knobloch 1992) in addition to Y. Proceeding as above one recovers (16), but with additional terms $-\mathrm{i}\nu c_g A^{\pm}_{Y'}$ on the right-hand side. Separable solutions of the form $A^{\pm}(Y, Y', \tau) = A^{\pm}(Y, \tau)\,\mathrm{e}^{\mathrm{i}nY'}$ then lead to (16) for $A^{\pm}(Y, \tau)$, with $\alpha_r = \nu n c_g \neq 0$, provided $n \neq 0$. As shown below, the detuning plays an important dynamical role because of the parametric forcing. In the edge-wave problem the imaginary parts of α, γ, $\varDelta$ and $\varLambda$ are of either dissipative or radiative origin, as discussed further in §5; in the absence of forcing these terms are responsible for the decay of the waves, as can be readily verified by constructing an equation for the wave energy $E \equiv \frac{1}{2}[\langle|A^+|^2\rangle_{\chi^+} + \langle|A^-|^2\rangle_{\chi^-}]$. The waves can only be maintained against this decay by the incident wavetrain ($\beta \neq 0$).

The presence of the mean field terms in (16*a*, *b*) is a consequence of the $O(1)$ group velocity. These terms imply that counterpropagating patterns 'see' each other only in the mean. These terms do not affect the stability properties of a unidirectional wavetrain, but do change the stability properties of standing wavetrains. This effect has been previously discussed in the context of water waves on arbitrary depth (Pierce & Knobloch 1993) and will be discussed further below. The averaged parametric excitation terms ($\beta\langle A^+\rangle_{\chi^+}$ and $\beta\langle A^-\rangle_{\chi^-}$) have not been previously analysed within the context of an O(2) symmetric problem with $O(1)$ group velocity. The most important effect of the averaging is to eliminate all spatial dependence in the excitation terms. Thus we find that parametric excitation can only couple *directly* to spatially uniform patterns. It should be emphasized that this does not preclude solutions which are not spatially uniform, but it does mean that spatially non-uniform wavetrains will be damped by radiation and so will decay.

For homogeneous wavetrains the evolution equations (16*a*, *b*) reduce to coupled complex Ginzburg–Landau equations

$$\mathrm{i}\nu A^+_\tau = \alpha A^+ + \beta A^- + (\varDelta + \Theta) A^+|A^+|^2 + \varLambda A^+|A^-|^2, \qquad (17a)$$

$$-\mathrm{i}\nu A^-_\tau = \bar{\alpha} A^- + \beta A^+ + (\bar{\varDelta} + \Theta) A^-|A^-|^2 + \bar{\varLambda} A^-|A^+|^2, \qquad (17b)$$

as dictated by the resulting O(2) symmetry of the system. For uniformly sloping beaches, equations of this form were derived by Akylas (1983) and Miles (1991). In the case of no dissipation (or radiation), all the coefficients on the right-hand side of (17*a*, *b*) are real; more generally, the coefficients α, $\varDelta$ and $\varLambda$ will be complex. In the general case, (17*a*, *b*) have been studied by Riecke *et al.* (1988), who showed that the parametric forcing ($\beta \neq 0$) stabilizes standing waves at onset, even in cases where, in the absence of parametric forcing, travelling waves would be the preferred pattern. The basic reason is that the parametric forcing couples the waves that propagate in the upshore and downshore directions, eliminating pure propagating waves (i.e. $A^+ = 0$ or $A^- = 0$) as solutions of the nonlinear problem. The counterpart of the propagating waves is a mixed state that bifurcates from the branch of standing waves at finite amplitude. See Riecke *et al.* (1988) for a detailed discussion of the secondary bifurcations described by (17*a*, *b*).

The mean flow plays two roles in the derivation of (16). First, it affects the values of the nonlinear coefficients. In the absence of such flows $\rho = 0$ and hence $\varDelta = \delta, \Theta = 0$ and $\varLambda = \lambda$. This change in the coefficients affects the stability properties of the various possible solutions, both in the parametrically forced system ($\beta \neq 0$) and in the unforced system ($\beta = 0$). For example, in the absence of dissipation, equations (16*a*, *b*) suggest that a free train of edge waves undergoes an instability leading to edge-wave solitons provided that $\varDelta\gamma > 0$, a condition that is affected by the value of ρ. However, the calculation leading to this result ignores the requirement that the associated mean flow satisfy equation (9*a*, *b*). Since these equations are incompatible with the soliton solution to equations (16*a*, *b*) we conclude that such solutions cannot be described within the inviscid shallow-water equations. The importance of the mean flow in predictions of this type was not recognized by Akylas (1983), who set all the [*n*00] terms to zero and consequently the coefficient ρ as well. In §4 we present explicit examples showing that the coefficients ρ, θ_1 and ν are all of $O(1)$, and hence that the coupling to the mean flow results in an $O(1)$ change in the nonlinear terms. Only for spatially homogeneous standing waves do the mean flow contributions cancel. It should be pointed out that for deep-water waves over a horizontal bottom of depth h_0, the corresponding mean flow coupling coefficient ρ remains finite for finite h_0 but vanishes as h_0^{-1} in the limit that the depth goes to infinity (Pierce & Knobloch 1993). Since the edge waves inhabit a region where the depth is always of $O(1)$, it seems likely that the coefficient ρ remains finite even in deep-water formulations of the edge-wave problem. In addition, the mean flow enters the stability analysis through (9) and (13). In particular, the perturbed mean flow must also satisfy the shallow-water wave equation (9*a*). This is an important constraint, and we will return to it in §5.

3. Stokes waves and modulational stability

We now consider various special solutions to (16*a*, *b*) for inhomogeneous wavetrains, and to (17*a*, *b*) for homogeneous wavetrains. We consider two cases, $\alpha_i = \Delta_i = 0$, $\Lambda_i < 0$ (radiative damping) and $\alpha_i < 0, \Delta_i < 0$, and $\Lambda_i < 0$ (linear and nonlinear damping as well as radiative damping). We focus first on spatially homogeneous wavetrains of the form

$$A^{\pm} = [R^{\pm}(\tau)]^{\frac{1}{2}} e^{\pm i\theta^{\pm}(\tau)}, \tag{18}$$

where $R^{\pm}$ and $\theta^{\pm}$ are real functions. Substituting these expressions into (17*a*, *b*) yields a set of four ordinary differential equations:

$$\nu R^{\pm}_{\tau} = 2\alpha_i R^{\pm} - 2\beta(R^+R^-)^{\frac{1}{2}} \sin(\theta^+ + \theta^-) + 2\Delta_i R^{\pm 2} + 2\Lambda_i R^+ R^-, \tag{19a}$$

$$\nu \theta^{\pm}_{\tau} = -\alpha_r - \beta(R^{\mp}/R^{\pm})^{\frac{1}{2}} \cos(\theta^+ + \theta^-) - (\Delta_r + \Theta) R^{\pm} - \Lambda_r R^{\mp}. \tag{19b}$$

In terms of the variables $R^{\pm} \equiv R \pm Q$ and $\theta^{\pm} \equiv \frac{1}{2}(\theta \pm \psi)$, these four equations become

$$\nu R_{\tau} = 2\alpha_i R - 2(R^2 - Q^2)^{\frac{1}{2}} \beta \sin\theta + 2\Delta_i(R^2 + Q^2) + 2\Lambda_i(R^2 - Q^2), \tag{20a}$$

$$\nu Q_{\tau} = 2(\alpha_i + 2\Delta_i R) Q, \tag{20b}$$

$$\nu \theta_{\tau} = -2\alpha_r - \frac{2R\beta\cos\theta}{(R^2 - Q^2)^{\frac{1}{2}}} - 2(\Delta_r + \Theta + \Lambda_r) R, \tag{20c}$$

$$\nu \psi_{\tau} = \frac{2Q\beta\cos\theta}{(R^2 - Q^2)^{\frac{1}{2}}} - 2(\Delta_r + \Theta - \Lambda_r) Q. \tag{20d}$$

Here the variable ψ represents the total phase and decouples completely from the system. This is a consequence of the translation invariance of the system in the longshore direction. The phase difference, θ, does not decouple, however, because the parametric forcing breaks time-translation invariance (cf. Crawford & Knobloch 1991). Equation (20*b*) demonstrates at once that in the presence of linear or nonlinear dissipation the variable Q evolves toward zero. Consequently, standing waves ($R^+ = R^-$) are the preferred waveform (cf. Riecke *et al.* 1988). In order to maintain the generality of the following discussion, we take Q to be a constant, bearing in mind that in the dissipative case $Q = 0$. Note that solutions with $Q \neq 0$ are accompanied by a mean flow $B_Y = 2\rho Q$.

The Stokes-wave solution to (20) is given by constant R, Q, θ, and $\psi_\tau \equiv \Omega$. To study its modulational stability properties, we linearize (16 a, b) around this solution. Accordingly, we set

$$A^{\pm} = (R \pm Q)^{\frac{1}{2}} \mathrm{e}^{\pm \mathrm{i}(\theta \pm \Omega\tau)/2}(1 + v^{\pm}(\chi^{\pm}, \tau)), \tag{21a}$$

$$B = B^+ + w^+(\chi^+, \tau) + B^- + w^-(\chi^-, \tau) + \kappa T. \tag{21b}$$

The perturbations to the mean flow, $w^{\pm}$, must in turn satisfy the linearized equation (9 b) *plus* the additional wave equation (9 a) as discussed in §5. The linearized equations for the edge-wave perturbations, $v^{\pm}$, are

$$\mathrm{i}\nu v^+_\tau = \gamma v^+_{\chi^+\chi^+} + c^+_1 v^+ + c^+_2 \overline{v^+} + c^+_3 \langle v^+ \rangle_{\chi^+} + c^+_4 \langle \overline{v^+} \rangle_{\chi^+} + c^+_5 \langle v^- \rangle_{\chi^-} + c^+_6 \langle \overline{v^-} \rangle_{\chi^-}, \tag{22a}$$

$$-\mathrm{i}\nu v^-_\tau = \bar{\gamma} v^-_{\chi^-\chi^-} + \overline{c^-_1} v^- + \overline{c^-_2}\, \overline{v^-} + \overline{c^-_3} \langle v^- \rangle_{\chi^-} + \overline{c^-_4} \langle \overline{v^-} \rangle_{\chi^-} + \overline{c^-_5} \langle v^+ \rangle_{\chi^+} + \overline{c^-_6} \langle \overline{v^+} \rangle_{\chi^+}, \tag{22b}$$

where

$$c^{\pm}_1 = \Delta R^{\pm} - \beta \mathrm{e}^{-\mathrm{i}\theta}(R^{\mp}/R^{\pm})^{\frac{1}{2}}, \quad c^{\pm}_2 = \Delta R^{\pm}, \tag{23a, b}$$

$$c^{\pm}_3 = \Theta R^{\pm}, \quad c^{\pm}_4 = \Theta R^{\pm}, \tag{23c, d}$$

$$c^{\pm}_5 = \Lambda R^{\mp} + \beta \mathrm{e}^{-\mathrm{i}\theta}(R^{\mp}/R^{\pm})^{\frac{1}{2}}, \quad c^{\pm}_6 = \Lambda R^{\mp}. \tag{23e, f}$$

The averages over spatially inhomogeneous perturbations are themselves homogeneous, thereby coupling the inhomogeneous perturbations to the amplitude and phase perturbations. This provides a potential mechanism for driving instabilities. Thus we consider perturbations of the form

$$v^{\pm} = \tilde{r}^{\pm}(\tau) + \tilde{s}^{\pm}(\tau)\mathrm{e}^{\mathrm{i}m^{\pm}\chi^{\pm}} + \overline{\tilde{t}^{\pm}(\tau)}\,\mathrm{e}^{-\mathrm{i}m^{\pm}\chi^{\pm}}. \tag{24}$$

Ordering the variables as $W = \{\tilde{r}^+, \overline{\tilde{r}^+}, \tilde{r}^-, \overline{\tilde{r}^-}, \tilde{s}^+, \tilde{t}^+, \overline{\tilde{s}^+}, \overline{\tilde{t}^+}, \tilde{s}^-, \tilde{t}^-, \overline{\tilde{s}^-}, \overline{\tilde{t}^-}\}^T$, the differential equations take the form

$$\nu W_\tau = \begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ 0 & \begin{pmatrix} \begin{pmatrix} \boldsymbol{M}^+_3 & 0 \\ 0 & \overline{\boldsymbol{M}^+_3} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \boldsymbol{M}^-_3 & 0 \\ 0 & \overline{\boldsymbol{M}^-_3} \end{pmatrix} \end{pmatrix} \end{pmatrix} W, \tag{25}$$

where $\boldsymbol{M}_1$ is a 4×4 matrix, $\boldsymbol{M}_2$ is 4×8, and $\boldsymbol{M}^{\pm}_3$ are 2×2. The characteristic equation for this system takes the following special form:

$$\det[\boldsymbol{M}_1 - \tilde{\lambda}\boldsymbol{I}]\det[\boldsymbol{M}^+_3 - \tilde{\lambda}\boldsymbol{I}]\det[\overline{\boldsymbol{M}^+_3} - \tilde{\lambda}\boldsymbol{I}]\det[\boldsymbol{M}^-_3 - \tilde{\lambda}\boldsymbol{I}]\det[\overline{\boldsymbol{M}^-_3} - \lambda\boldsymbol{I}] = 0, \tag{26}$$

where $\tilde{\lambda}$ is the eigenvalue. It follows that the stability criteria for amplitude perturbations ($\tilde{r}$) and modulation perturbations ($\tilde{s}, \tilde{t}$) decouple completely. In the following subsections, we consider these in turn.

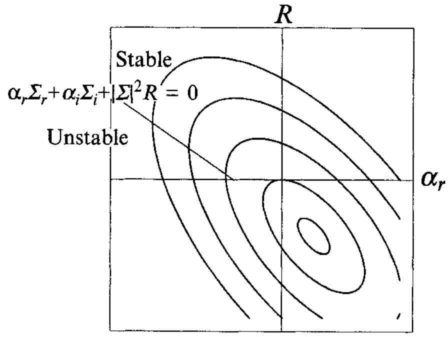

FIGURE 1. Bifurcation diagram for standing waves ($\alpha_i < 0$). The concentric ellipses are the loci of solutions for various values of $\beta^2 = \{0.1, 1, 3, 6, 10\}$. All states with $R < 0$ are unphysical, while all states below the diagonal line are unstable to spatially homogeneous perturbations. The coefficient values are: $\alpha_i = -1$, $\Sigma = 1-1.22\mathrm{i}$.

3.1. *Homogeneous perturbations*: $\alpha_i < 0$ *or* $\Delta_i < 0$

The matrix $\boldsymbol{M}_1$ governing the stability of uniform wavetrains with respect to homogeneous perturbations is given by

$$\boldsymbol{M}_1 = -\mathrm{i}\begin{pmatrix} (c_1^+ + c_3^+) & (c_2^+ + c_4^+) & c_5^+ & c_6^+ \\ -(\overline{c_2^+} + \overline{c_4^+}) & -(\overline{c_1^+} + \overline{c_3^+}) & -\overline{c_6^+} & -\overline{c_5^+} \\ -\overline{c_5^-} & -\overline{c_6^-} & -(\overline{c_1^-} + \overline{c_3^-}) & -(\overline{c_2^-} + \overline{c_4^-}) \\ c_6^- & c_5^- & (c_2^- + c_4^-) & (c_1^- + c_3^-) \end{pmatrix}. \tag{27}$$

There are two cases to consider, the dissipative case with $R^+ = R^- = R$, and the case with neither linear nor nonlinear dissipation for which solutions with $R^+ \neq R^-$ become possible. The former case is considered in this section, while the latter is postponed to §3.2. In the present case, either linear ($\alpha_i \leqslant 0$) or nonlinear ($\Delta_i \leqslant 0$) damping is sufficient to force $Q = 0$ as the only solution to (20*b*). The remaining equations (20) have the non-trivial solution:

$$Q = 0, \quad \psi = 0, \quad \theta = -\tan^{-1}\left(\frac{\alpha_i + \Sigma_i R}{\alpha_r + \Sigma_r R}\right), \tag{28a–c}$$

$$R = |\Sigma|^{-2}\{-(\alpha_r\Sigma_r + \alpha_i\Sigma_i) \pm [(\alpha_r\Sigma_r + \alpha_i\Sigma_i)^2 - |\Sigma|^2(|\alpha|^2 - \beta^2)]^{\frac{1}{2}}\}, \tag{28d}$$

where $\Sigma \equiv \Delta + \Theta + \Lambda$. This solution corresponds to a spatially uniform standing wave provided (28*d*) defines a real positive value of R. This occurs whenever

$$\beta^2 > |\alpha|^2. \tag{29}$$

Equation (29) implies that observable edge waves require that the strength of the parametric forcing exceeds the linear damping, and must be even larger if it is off-resonance ($\alpha_r \neq 0$). Neither radiation damping nor nonlinear damping plays a role in the bifurcation from the trivial state, though both control the saturation amplitude of the resulting standing waves. We now see that a pure edge-wave mode will exist in one of two possible circumstances. Either the forcing frequency (or alternatively, the longshore wavenumber) may be restricted to a band where only a single mode exists, or if multiple modes are allowed by the linear problem, parameters must be chosen such that exactly one of the modes gives a positive value of R.

Figure 1 shows the resulting bifurcation diagram in the (α_r, R)-plane. The standing-wave solutions take the form of a family of ellipses parametrized by β^2; the solutions with $R < 0$ are not physically realizable. The figure and the following discussion are based on the assumption that $\Sigma_r > 0$; if $\Sigma_r < 0$, the ellipses tilt the opposite way and α_r should be replaced by $-\alpha_r$ in the discussion below. The figure shows that the detuning parameter, α_r, is dynamically significant. If $|\alpha_r|$ is too large there can be no solutions. As α_r increases from a large negative value, a standing edge wave bifurcates from the trivial solution at $\alpha_r = -(\beta^2-\alpha_i^2)^{\frac{1}{2}}$; the trivial solution is unstable to standing edge waves for $-(\beta^2-\alpha_i^2)^{\frac{1}{2}} < \alpha_r < (\beta^2-\alpha_i^2)^{\frac{1}{2}}$, while for $\alpha_r > (\beta^2-\alpha_i^2)^{\frac{1}{2}}$ there are again no solutions. For sufficiently large β^2, there are two standing wave solutions in the interval $(\alpha_i \Sigma_r + |\Sigma|\,|\beta|)/\Sigma_i < \alpha_r < -(\beta^2-\alpha_i^2)^{\frac{1}{2}}$, with the two branches annihilating at a secondary saddle-node bifurcation as α_r becomes increasingly negative. This picture (with $\alpha_i \equiv 0$) has been previously discussed by Rockliff (1978), and is also familiar from studies of the Faraday system.

The stability assignments follow either from the matrix $\boldsymbol{M}_1$, or more simply by noting that since the system (20) is stable with respect to perturbations in Q, the stability problem has only two significant eigenvalues. These correspond to perturbations $\hat{r}(\tau)$ and $\hat{t}(\tau)$ in the amplitude, R, and phase, θ, respectively:

$$\nu\begin{pmatrix}\hat{r}\\ \hat{t}\end{pmatrix}_\tau = 2\begin{pmatrix}\Sigma_i R & (\alpha_r+\Sigma_r R)\,R\\ -\Sigma_r & (\alpha_i+\Sigma_i R)\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{t}\end{pmatrix}. \tag{30}$$

The eigenvalues of the coefficient matrix are

$$\alpha_i+2\Sigma_i R \pm [(\alpha_i+2\Sigma_i R)^2-4R(\alpha_r \Sigma_r+\alpha_i \Sigma_i)-4|\Sigma|^2R^2]^{\frac{1}{2}}. \tag{31}$$

Since $\alpha_i < 0$, $\Sigma_i < 0$ and $R > 0$ we have stability if and only if

$$\alpha_r \Sigma_r+\alpha_i \Sigma_i+|\Sigma|^2 R \geqslant 0.$$

The point $\alpha_r \Sigma_r+\alpha_i \Sigma_i+|\Sigma|^2 R = 0$ corresponds to the secondary saddle-node bifurcation. From figure 1 it may be seen that in the interval of α_r where there are two solutions, the solution with the larger value of R is stable, while the smaller solution is unstable. The analysis above shows that there are no other instabilities with respect to spatially homogeneous perturbations.

3.2. *Homogeneous perturbations:* $\alpha_i = \Delta_i = 0$

In the case of no linear or nonlinear damping ($\alpha_i = \Delta_i = 0$) but with radiation damping ($\Lambda_i < 0$), which is appropriate for inviscid shallow-water edge waves, solutions with $R^+ \neq R^-$ become possible. Such solutions correspond to mixed modes. These are two-frequency propagating waves, and differ from pure travelling waves in lacking the symmetry of a rotating wave; i.e. for the mixed modes, evolution in time is no longer equivalent to spatial translations. These waves are given by the one-parameter family of solutions

$$\theta = \tan^{-1}\left(\frac{-\Lambda_i R}{\alpha_r+\Sigma_r R}\right), \quad Q = R\left(1-\frac{\beta^2}{|\alpha+\Sigma R|^2}\right)^{\frac{1}{2}}, \quad \psi = \frac{-2Q}{\nu R}[\alpha_r+2(\Delta_r+\Theta)\,R]\,\tau \tag{32a–c}$$

for all R such that $|\alpha+\Sigma R| > |\beta|$, and are the counterparts of the generalized Stokes waves in the water-wave problem studied by Pierce & Knobloch (1993), constrained to be spatially homogeneous so that they can couple to the spatially averaged parametric excitation. Depending on the value of Q, they represent either standing waves ($Q = 0$) or an asymmetric mixture of left- and right-travelling waves. There are no pure travelling waves ($Q = \pm R$) unless there is no parametric forcing ($\beta = 0$).

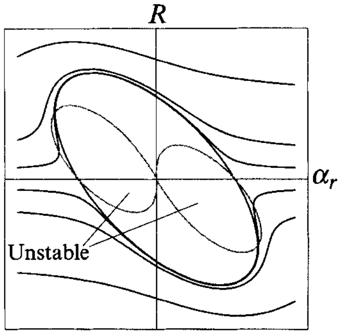

FIGURE 2. Bifurcation diagram for $\alpha_i = 0$. The large ellipse is the locus of standing waves, $Q = 0$. The remaining black curves are the loci of travelling waves at $Q = \{0.01, 0.1, 0.9\}$. All states with $R < 0$ are unphysical, while all states in the interior of the grey curve are unstable to spatially homogeneous perturbations. The coefficient values are: $\alpha_i = 0$, $\Sigma = 1-1.22\mathrm{i}$, $\beta^2 = 2$.

Since the right-hand side of (20*b*) is identically zero, perturbations to Q are associated with a zero eigenvalue. Consequently, only perturbations $\hat{r}$ and $\hat{t}$ in the remaining two variables R and θ, respectively, must be considered. The resulting system is

$$\nu\begin{pmatrix}\hat{r}\\ \hat{t}\end{pmatrix}_T = 2\begin{pmatrix} \Lambda_i R & \dfrac{R\beta^2}{|\alpha+\Sigma R|^2}[\alpha_r+\Sigma_r R] \\ R^{-1}\left(\alpha_r-[\alpha_r+\Sigma_r R]\dfrac{|\alpha+\Sigma R|^2}{\beta^2}\right) & \Lambda_i R \end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{t}\end{pmatrix}. \tag{33}$$

The eigenvalues of the coefficient matrix are

$$2\Lambda_i R \pm 2\left(\alpha_r[\alpha_r+\Sigma_r R]\frac{\beta^2}{|\alpha+\Sigma R|^2}-[\alpha_r+\Sigma_r R]^2\right)^{\frac{1}{2}}. \tag{34}$$

Using these results, it is now simple to examine the stability of waves with a given Q as a function of their amplitude R, or of waves with different Q as a function of the detuning parameter, α_r. Figure 2 shows the result of such an analysis. The figure shows the bifurcation diagram for $\alpha_i = \Lambda_i = 0$, with the standing and travelling wave states as functions of α_r and R for various values of Q and a fixed β. States with $R < 0$ are not physically realizable. The large ellipse centred on the origin is the locus of standing-wave solutions ($Q = 0$). The exterior of this ellipse is completely filled with travelling wave states ($Q > 0$); no spatially homogeneous solutions exist in the interior. All solutions lying in the interior of the grey curve are unstable to homogeneous perturbations, since in this region (34) yields a real positive eigenvalue. Thus the segment of standing waves between the saddle-node bifurcation and $R = 0$ is again unstable, and there is an additional region of unstable travelling waves adjacent to the unstable branch of standing waves. The corresponding segment of the grey curve is in fact the locus of secondary saddle-node bifurcations occurring on the travelling wave curves. Each travelling wave curve which intersects the grey curve loses stability at a saddle-node upon entering the region, and regains stability at a saddle-node upon leaving it. For these curves, there is consequently an interval of α_r such that $R(\alpha_r)$ is triple valued, with the middle branch being unstable to spatially homogeneous perturbations and the other two branches being stable. There are no other instabilities with respect to spatially homogeneous perturbations. As a consequence, variation of the parameter α_r creates a mechanism for breaking the reflection symmetry of the solutions. For any given β^2, a sufficiently large value of $|\alpha_r|$ forces the system off the ellipse of standing waves and into the travelling wave states. However, it seems unlikely that these solutions could be observed experimentally because of the dominant effect of viscous dissipation on laboratory lengthscales, which will always cause the travelling wave states to decay (cf. §3.1).

3.3. *Inhomogeneous perturbations*

We now return to the stability matrix in (25) and discuss the stability of uniform wavetrains with respect to spatially varying perturbations, i.e. those with $\tilde{r} = 0$, but $\tilde{s} \neq 0$ and $\tilde{t} \neq 0$. The matrices $\boldsymbol{M}_3^{\pm}$ governing modulational stability are

$$\boldsymbol{M}_3^+ = \mathrm{i}\begin{pmatrix} -(-\gamma m^{+2} + c_1^+) & -c_2^+ \\ \overline{c_2^+} & (-\bar{\gamma} m^{+2} + \overline{c_1^+}) \end{pmatrix}, \tag{35a}$$

$$\boldsymbol{M}_3^- = \mathrm{i}\begin{pmatrix} (-\bar{\gamma} m^{-2} + \overline{c_1^-}) & \overline{c_2^-} \\ -c_2^- & -(-\gamma m^{-2} + c_1^-) \end{pmatrix}. \tag{35b}$$

The eigenvalues of these matrices are

$$c_{1i}^+ - \gamma_i m^{+2} \pm [|c_2^+|^2 - (c_{1r}^+ - \gamma_r m^{+2})^2]^{\frac{1}{2}}, \tag{36a}$$

$$c_{1i}^- - \gamma_i m^{-2} \pm [|c_2^-|^2 - (c_{1r}^- - \gamma_r m^{-2})^2]^{\frac{1}{2}}. \tag{36b}$$

There are now four conditions which must all be satisfied for modulational stability of a spatially homogeneous Stokes wave solution:

$$c_{1i}^{\pm} - \gamma_i m^{\pm 2} \leqslant 0 \quad \text{and} \quad |c_2^{\pm}| \leqslant |c_1^{\pm} - \gamma m^{\pm 2}| \tag{37a, b}$$

for all real $m^{\pm}$. The latter condition depends strongly on the dispersive character of the equations (i.e. on the real parts of the coefficients). However, provided that $\gamma_i > 0$, both inequalities are always satisfied for sufficiently large $m^{\pm}$, so that instability is restricted to finite wavenumbers. Consequently, the latter condition will yield instability if and only if the following quadratic equation for $m^{\pm 2}$ has a real positive root:

$$|\gamma|^2 m^{\pm 4} - 2(c_{1r}^{\pm}\gamma_r + c_{1i}^{\pm}\gamma_i) m^{\pm 2} + |c_1^{\pm}|^2 - |c_2^{\pm}|^2 = 0. \tag{38}$$

The conditions for modulational stability of the Stokes waves are therefore

$$c_1^{\pm} \leqslant 0 \quad \text{and} \quad |\gamma|^2(|c_1^{\pm}|^2 - |c_2^{\pm}|^2) > (\max\{0, (c_{1r}^{\pm}\gamma_r + c_{1i}^{\pm}\gamma_i)\})^2. \tag{39a, b}$$

The 'max' is sufficient to handle the cases of two imaginary roots versus two real negative roots. It follows from (20*a*) that $\beta \sin\theta < 0$, so that the first condition is always satisfied for Stokes waves. The second condition depends in detail upon the actual numerical values of the coefficients, but we note that in the absence of dissipation and parametric forcing we regain the Benjamin–Feir criterion for the stability of free wavetrains: $\Delta\gamma < 0$. It is important to note that (39*b*) depends on the mean flow (i.e. on ρ) *even* in the case of standing waves. Furthermore, these conditions are contingent on the existence of a compatible mean flow in the bulk satisfying (9*a, b*). We also emphasize that the stability requirements (39) apply to inhomogeneous perturbations only. Note in particular that, as $m^{\pm} \to 0$, the stability conditions (37) do not reduce to those governing stability with respect to spatially homogeneous perturbations (cf. Knobloch 1992; Pierce & Knobloch 1993). This discontinuity is a consequence of the mean field coupling terms. In the next section we compute the coefficients of (15) for two different beach profiles and use the above results to discuss the stability of the resulting edge waves.

4. Numerical results: exponential and linear profiles

For essentially all relevant profiles, it is either inefficient or impossible to actually solve for the spectra of equations (7) and evaluate the integral transforms to obtain the particular solutions. It is much easier to discretize the equations and solve them numerically. The first step is to solve the linear eigenvalue problem, followed by the higher-order boundary-value problems. The solvability criteria may be evaluated and the coefficients of the amplitude equations obtained straightforwardly by singular-value decomposition of the discretized matrix operator (Mahalov & Leibovich 1992) rather than by evaluating the integrals in (11). In particular, the left null vector of the operator matrix corresponds to the discretized adjoint solution to within appropriate quadrature weights. The inner product of the left null vector and the discretized inhomogeneous term then approximates the continuous integral solvability criterion.

We have reduced the computational domain from semi-infinite to compact by assuming that the depth profile is constant in the bulk for $x \geqslant x_0$. Thus we know the exact solutions up to constants of integration, and we may impose exact boundary conditions at any $x_1 \geqslant x_0$. The two offshore-propagating solutions, j and q, are potentially complex and boundary conditions must be chosen to satisfy the appropriate conditions on the external wave field. The solution j is a superposition of incident and reflected waves, but the absence of viscosity and surface tension from the linear problem requires total reflection (Miles 1990*a*, *c*). Thus j is in fact real up to an overall normalization constant, which we choose to give j a convenient normalization: $j^2+(j'/2\omega)^2 = 1/\pi$. The solution q has a particular solution asymptotic to a radiated wave, $e^{2i\omega x}$, but no incident component. The boundary conditions at x_1 for p, q and r may be found by multiplying (6*a*), (7*d*) and (7*f*) by e^{-Kx}, $e^{2i\omega x}$ and e^{-2Kx}, respectively, and integrating by parts over $[x_1, \infty]$ subject to the assumption that $\{pe^{-Kx}, p'e^{-Kx}, -2i\omega q+q', re^{-2Kx}, r'e^{-2Kx}\} \rightarrow 0$ as $x \rightarrow \infty$. The boundary conditions imposed at x_1 are thus

$$Ka+a' = 0, \quad Kp+p' = a(k-c_g\,\omega)/K, \tag{40\textit{a, b}}$$

$$-2i\omega q+q' = \frac{\omega a^2}{K-i\omega}(4k^2-3\omega^2), \quad 2Kr+r' = \frac{-3\omega^3 a^2}{4K}, \tag{40\textit{c, d}}$$

where $K = (k^2-\omega^2)^{\frac{1}{2}}$. Note that there are also non-trivial boundary conditions on $e^{\pm}$ (at order $[31(\pm 1)]$) at x_1 which cannot be ignored. These are derived from (8*b*, *c*) by multiplying by e^{-Kx} and integrating over $[x_1, \infty]$, after noting that each second-order solution may be written explicitly in this domain up to a single constant of integration. In general, these boundary conditions affect the nonlinear coefficients, but the contribution falls off with increasing k and fixed x_1 due to the exponential decay of a. For the examples discussed below, the contributions are significant only for small k.

The problems were discretized by Chebyshev collocation on the interval $[0, x_1]$. The mesh points are defined by $\{x = \frac{1}{2}x_1(1-\cos(\pi(n-1)/(N-1)))\,|\, n = 1, \ldots, N\}$. Since $x = 0$ is a regular singular point with zero exponents for each equation, there is a regular and a singular solution. However, no explicit regularity condition is required since the singular solution is not representable by Chebyshev polynomials. Thus we need only $(N+1)$ polynomials for a well-posed representation of each problem except the one for

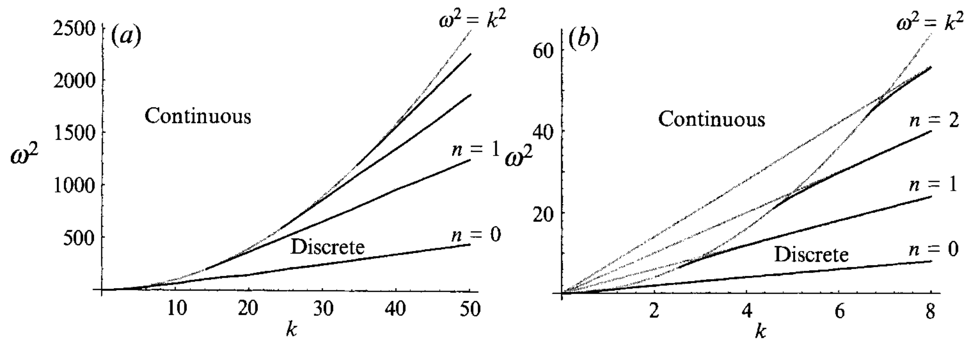

FIGURE 3. Discrete dispersion relations, ω^2 *vs.* $k\,(g = h_0 = 1)$, for (*a*) exponential and (*b*) piecewise-linear beach profiles. The grey parabolas, $\omega^2 = k^2$, are the boundary between continuous and discrete modes. The linear asymptotes are the dispersion relations for an unbounded linear profile (Rockliff 1978).

j. There is exactly one homogeneous solution for j which is bounded at $x = 0$, so that the discretized operator matrix for j is $N \times N$ and has rank $(N-1)$. The null vector of this operator matrix is the discretization of the homogeneous solution for both (7*c*) and (7*d*), but it does not satisfy the homogeneous boundary conditions for q (i.e. the radiation condition). In fact, there is no regular homogeneous solution to the combined equations for q, (7*d*) and (40*c*), so that the $(N+1) \times (N+1)$ operator matrix for q has full rank and is directly invertible.

We have performed the calculations for two experimentally relevant depth profiles:

$$\text{exponential:} \quad h(x) = 1 - \mathrm{e}^{-10x}, \tag{41a}$$

$$\text{piecewise linear:} \quad h(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ 1 & \text{for} \quad x \geqslant 1. \end{cases} \tag{41b}$$

The results for any scaling of x may be obtained by rescaling:

$$\{\hat{x}, \hat{a}, \hat{p}, \hat{q}, \hat{r}, \hat{k}, \hat{\omega}, \hat{c}_g, \hat{\rho}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\lambda}, \hat{\theta}_1, \hat{\theta}_2, \hat{\nu}, \hat{\mu}\}$$
$$\rightarrow \{xl, a, pl, q/l, r/l, k/l, \omega/l, c_g, \rho/l^2, \beta/l^2, \gamma l, \delta/l^3, \lambda/l^3, \theta_1/l, \theta_2/l^3, \nu, \mu/l\}. \tag{42}$$

In both cases we used $N = 60$, and set the outer boundary condition at $x_1 = 3$ and $x_1 = 1$ for the exponential and linear profiles, respectively. The exponential profile satisfies the saturation assumption to better than thirteen digits at x_1. For the linear profile, placing the boundary at the discontinuity in the beach slope enhances accuracy by pinning the discontinuity relative to the discretization mesh, as well as allowing super-linear convergence of the polynomial coefficients. The results of the linear eigenvalue problem are summarized in figure 3. The region above the curve $\omega^2 = k^2$ is the continuous spectrum. Below this curve, there exists a finite number of discrete modes for any given k. The most important result from the linear computations is that there exists a finite value of $k \equiv k_n, n \geqslant 1$, below which each mode no longer exists. Only the fundamental mode exists for all k. The critical wavenumbers for the first three modes are {14.1421, 24.4949, 34.6410} for the exponential profile and {2.52618, 4.56310, 6.57354) for the linear one. These values were found by locating singularities of the discretized operator matrix along $\omega^2 = k^2$. The grey lines in figure 3(*b*) are the dispersion relations for an unbounded linear profile within the shallow-water equations (Rockliff 1978); as expected, the dispersion relations for the piecewise-linear profile asymptote to these lines in the limit of large k.

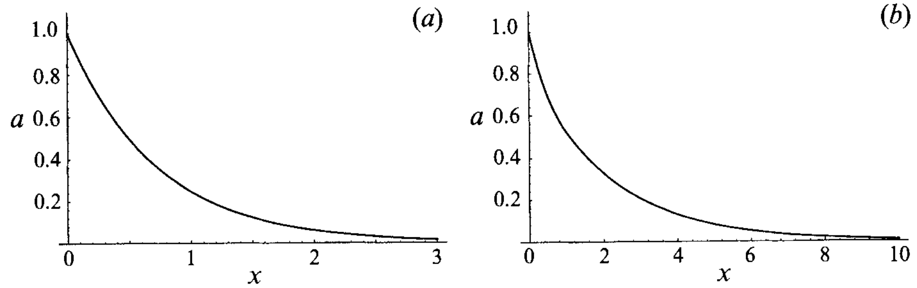

FIGURE 4. Linear eigenfunctions, $a(x)$: (a) exponential, $k = 4$; (b) linear, $k = 1$.

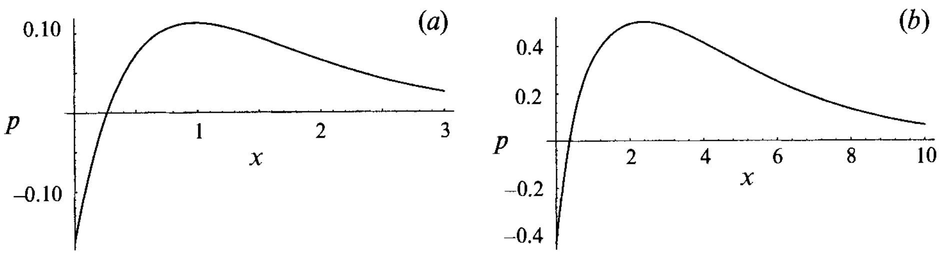

FIGURE 5. [211] mode, $p(x)$: (a) exponential, $k = 4$; (b) linear, $k = 1$.

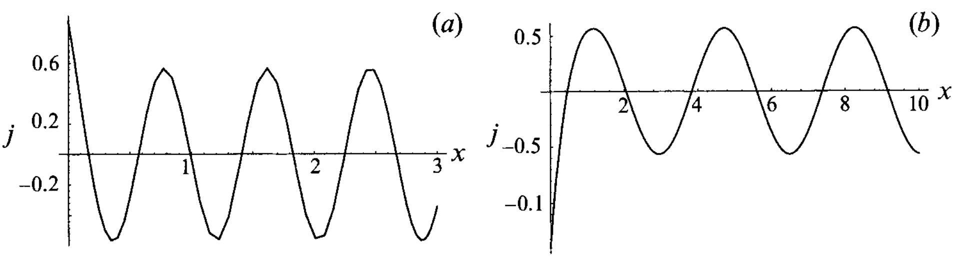

FIGURE 6. [202] homogeneous mode, $j(x)$: (a) exponential, $k = 4$; (b) linear, $k = 1$.

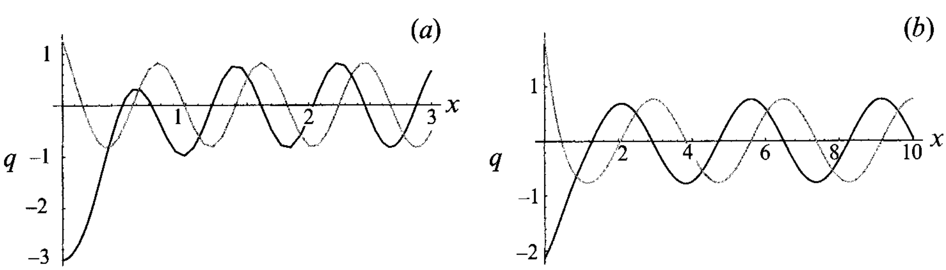

FIGURE 7. [202] inhomogeneous mode, $q(x)$. Black is the real part, grey the imaginary part. (a) Exponential, $k = 4$; (b) linear, $k = 1$.

For all wavenumbers (or frequencies) less than k_1, the single mode restriction is satisfied automatically even in the absence of linear damping. Figures 4–8 are plots of the solutions to (5), (6a) and (7) for both profiles at a representative $k < k_1$. The jaggedness of the oscillatory solutions does not reflect inaccuracy in the numerics, only the coarseness of the mesh. The oscillatory solutions show convergence after 30–40 polynomials, while non-oscillatory solutions converge after 10–20. Figures 9–16 are

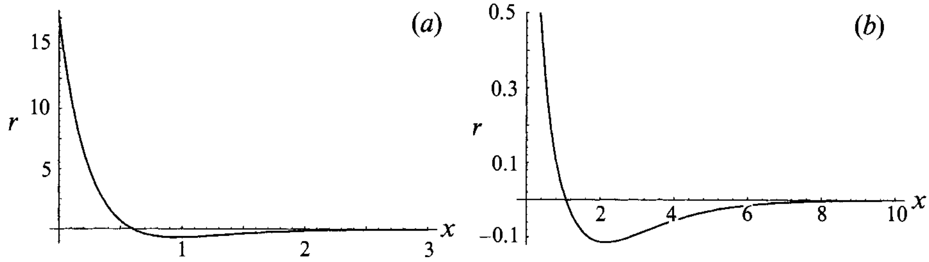

FIGURE 8. [222] mode, $r(x)$: (*a*) exponential, $k = 4$; (*b*) linear, $k = 1$.

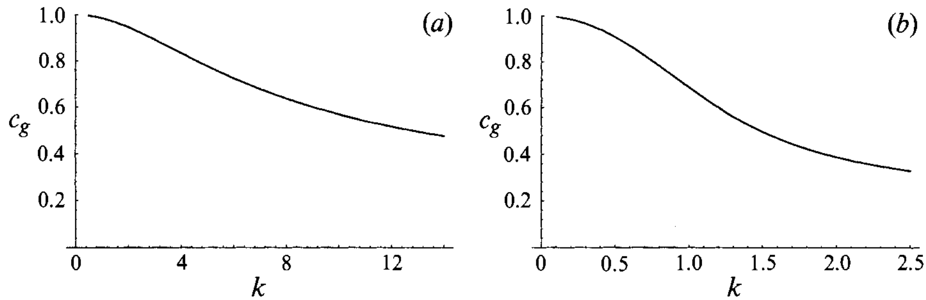

FIGURE 9. Group velocity for the fundamental mode: (*a*) exponential; (*b*) linear.

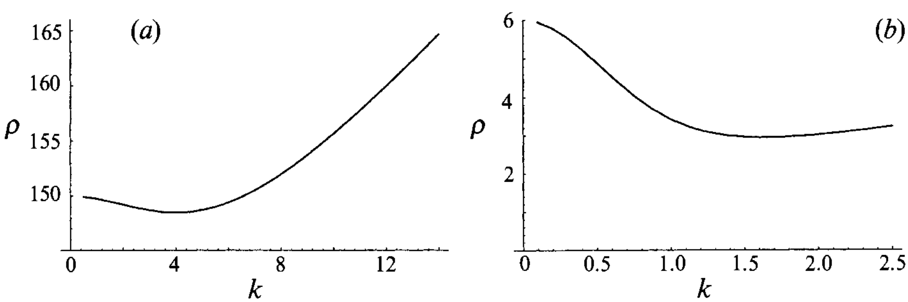

FIGURE 10. ρ for the fundamental mode: (*a*) exponential; (*b*) linear.

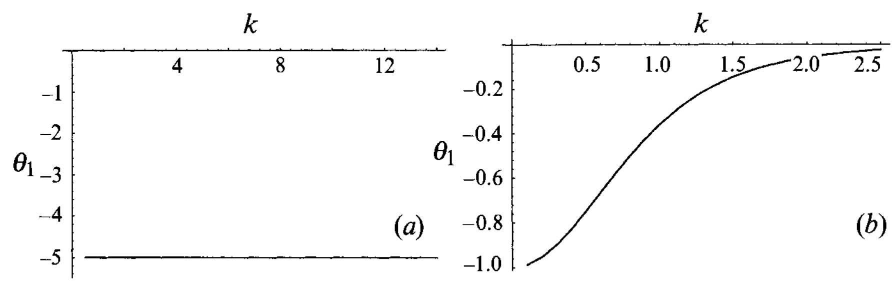

FIGURE 11. θ_1 for the fundamental mode: (*a*) exponential; (*b*) linear.

plots of the group velocity and amplitude equation coefficients as functions of k through the pure mode range, $0 < k < k_1$. The quantity $(\delta_r + \lambda_r)/(\delta_i + \lambda_i)$ tends to the value -0.82 for large k, in agreement with Rockliff (1978) for a linear unbounded profile. Note that the left null vector, which is the discretization of the linear

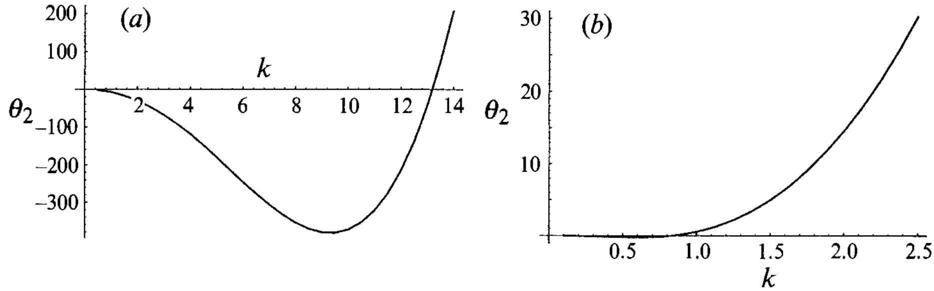

FIGURE 12. θ_2 for the fundamental mode: (*a*) exponential; (*b*) linear.

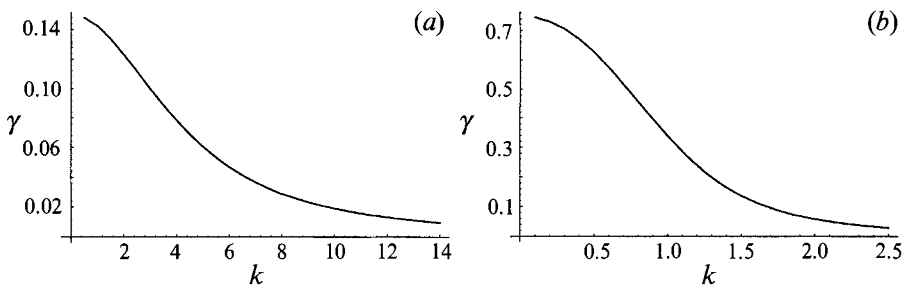

FIGURE 13. γ for the fundamental mode: (*a*) exponential; (*b*) linear.

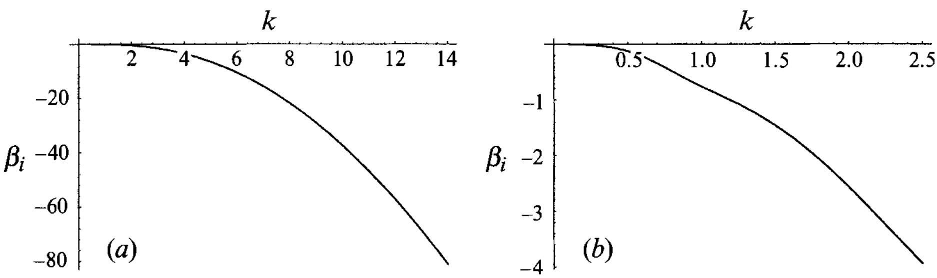

FIGURE 14. Imaginary part of β for the fundamental mode; the real part is zero: (*a*) exponential; (*b*) linear.

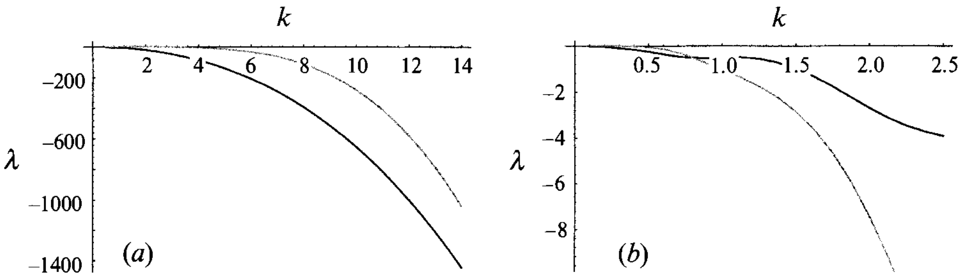

FIGURE 15. λ for the fundamental mode. Black is the real part, grey the imaginary part. (*a*) Exponential; (*b*) linear.

homogeneous adjoint solution, is only defined up to an arbitrary normalization. This normalization appears in both the plots and the evolution equations as an arbitrary function of k multiplying each of the nonlinear coefficients except ρ. In order to maintain the normalization assumed in (11), it is necessary to specify the normalization

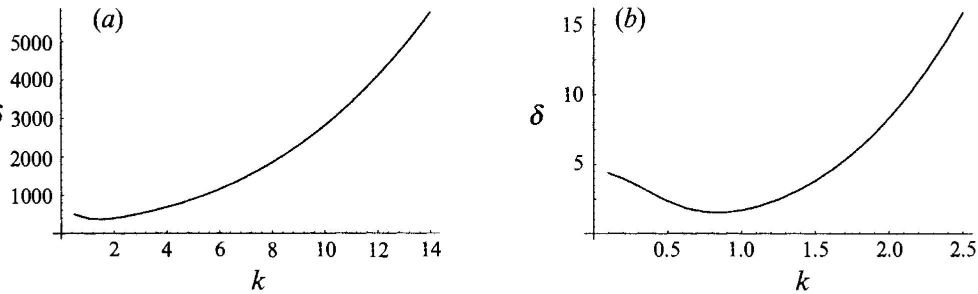

FIGURE 16. δ for the fundamental mode: (*a*) exponential; (*b*) linear.

of the left null vector appropriately. This is accomplished by setting a convenient normalization during the computations, evaluating $\int_0^\infty a^2\,\mathrm{d}x$ using the numerical solution for a and then rescaling the coefficients appropriately.

For edge waves in the shallow-water formulation with no detuning,

$$\alpha_r = \alpha_i = \Delta_i = \gamma_i = 0.$$

In this case, the expressions for the coefficients relevant to modulational stability reduce to $c_1^\pm = (\Delta R^\pm + \Sigma R^\mp)$ and $c_2^\pm = \Delta R^\pm$. If we specialize to standing waves ($R^+ = R^-$), we may drop the superscripts, and the stability condition (39*b*) is independent of R and hence of β as well. The numerical data can then be used to show that $c_{1r}^2 > (|c_1|^2 - |c_2|^2) > 0$ and that $c_{1r} > 0$ for both profiles. Furthermore, while our calculations span only a finite range of k, there is nothing in the data to suggest that either of these conditions will change upon consideration of a larger interval. Consequently, we find that the stability condition (39*b*) reduces to $\gamma_r < 0$, which is never the case for either profile. If it were possible to generate a compatible mean flow solution to (9*a*, *b*), we would then conclude that standing waves for both profiles are unstable to modulational perturbations.

We note that there appears to be a minor problem with our discretization method in the neighbourhood of $k = 10.5$ for the r problem with an exponential profile. For $N = 60$, the solution appears to be oscillatory instead of decaying, and the corresponding value of δ is too small by roughly 25 %. The problem persists to at least $N = 100$, but is eliminated by *reducing* the order to $N = 40$. This is more than sufficient to ensure accuracy for a decaying solution. We perform the inversion of the operator matrix by singular-value decomposition, and we hypothesize that by increasing the order we are introducing a singular vector which happens to be resonant with the discretized inhomogeneity. We emphasize that this is an artifact of the numerics; there are no homogeneous solutions for r which may be physically resonant.

5. Discussion

The primary concern of this paper has been the derivation of the correct evolution equations for spatially modulated counterpropagating edge waves in shallow water. We have shown that the class of Stokes wave solutions is more restricted than previous work would imply. This is due to the non-local nature of the equations, which eliminates the coupling of the parametric forcing to spatially inhomogeneous solutions. If linear viscous damping is included, only homogeneous standing waves remain as stable solutions. If the formulation is inviscid, there exists a one-parameter family of mixed homogeneous travelling waves. Pure travelling waves are never solutions of the parametrically driven problem. Spatially homogeneous pure travelling waves are, however, solutions of the unforced problem but only in the absence of linear damping. The work of Mathew & Akylas (1990) indicates that this is an artifact of the shallow-water formulation, since radiation is generated only via interactions between the counterpropagating wavetrains. In the more general finite-depth formulation one generically expects that quadratic self-interactions also generate radiation leading to nonlinear damping of the pure travelling waves as well.

The use of the inviscid shallow-water equations entails in addition a physical inconsistency related to the energy balance. With these equations the only non-singular solution for $j(x)$ corresponds to total reflection of the incident wave (cf. Miles 1990*a*, *c*). However, in general $q(x)$ has a non-trivial solution which corresponds to radiation driven by quadratic interactions of the counterpropagating edge waves. This leads to the apparent inconsistency that more energy is exiting the edge-wave system via the reflected and radiated waves than is entering with the incident wavetrain. Formally, there is no contradiction in the limit of small ϵ since an $O(\epsilon)$ edge wave can leak $O(\epsilon^2)$ radiation on timescales of $O(1/\epsilon^2)$ with only an $O(\epsilon^2)$ energy loss (energy densities and fluxes are proportional to the amplitude of the wave squared). This is in fact the origin of the dissipative radiation terms in (16*a*, *b*), since this $O(\epsilon^2)$ change in edge-wave energy is manifested as an $O(1)$ change in the amplitudes, $A^{\pm}$, on the $O(1/\epsilon^2)$ evolution timescale. This argument does not, however, provide a complete resolution to the question of how the edge wave may achieve an $O(\epsilon)$ amplitude while radiating and in the face of total reflection of the incident wavetrain. The question of energy balances in the initial value problem is likely to be resolved only through a complete accounting of viscous dissipation, surface tension, and contact line dynamics. This type of calculation would self-consistently allow partial reflection of the incident wavetrain (Miles 1990*a*, *c*) but is beyond the scope of this paper. It should be emphasized that total reflection of the incident wave is not built into the evolution equations; (16*a*, *b*) should result from any set of model equations which respect the inherent O(2) symmetry of the problem.

We have studied two experimentally relevant profiles, exponential and piecewise linear. The results for any rescaling of these profiles follow from (42). The primary linear result is that there exists a minimum forcing frequency and corresponding minimum wavenumber of the edge waves below which only the fundamental mode exists. Consequently there are intervals of ω and k such that the linear mode is unique. This is very important since in the absence of viscosity all linear modes bifurcate from the trivial state simultaneously. Formally, any asymptotic expansion that fails to include all allowed modal interactions is incomplete. Several authors have used unbounded depth profiles within the context of the shallow-water equations. This formally contradicts the basic assumptions of the governing equations, but it is prima facie acceptable since the evanescent edge wave falls off exponentially while the depth is typically assumed to increase linearly. However, the unbounded problem generically has a countably infinite discrete spectrum of evanescent modes for all forcing frequencies. Therefore, no asymptotic expansion with a finite number of linear modes can correctly represent the dynamics unless linear damping is included to split the bifurcation.

There are several ways to include linear viscous damping in a surface water wave problem. The most satisfactory method is to solve the Navier–Stokes equations for the full depth-dependent problem. While there has been some work on the linear aspects of this problem for specialized beach profiles, the weakly nonlinear problem for general beach profiles has not been attempted. The next best approach is to approximate the effects of weak dissipation through matched asymptotic expansion in the surface and bottom oscillatory boundary layers. However, there are several significant difficulties with this approach when applied to shallow-water edge waves. First, the shallow-water equations are not compatible with the assumption of oscillatory boundary layers which are asymptotically thin relative to the depth, unless the viscosity is assumed to scale with some power of the asymptotic parameter, ϵ. Even then, the contact line at $x = 0$ cannot be rigorously accounted for within this type of model. Excluding the problems at the water line, viscosity is still a singular perturbation in the Navier–Stokes equations; consequently this approach, which removes the viscosity from the linear problem and moves it to higher orders in the perturbation expansion, cannot be considered valid for more than a heuristic understanding of the effects of damping. A further problem with the boundary-layer approach arises with the mean flow. In effect, the linear damping must be included separately for each frequency of oscillation in the problem. This approach then implies that the mean flow must remain undamped since it does not oscillate on the fast timescale. Nonetheless, we accept the qualitative implications of boundary-layer damping, which leads us to consider a non-zero value of the coefficient $\alpha_i < 0$. While it is possible to calculate an expression for this coefficient within the context of the Boussinesq equations, this results in a considerable and not particularly enlightening increase in the complexity of the asymptotic expansion.

The shallow-water equations give $\gamma_i \equiv 0$, since there is no radiation generated by the Sturm–Liouville problem, (6*a*), and p is inherently real. In the generic formulation, we do not exclude the possibility that $\gamma_i > 0$, representing viscous dissipation on $O(\epsilon^{-1})$ scales. Note, however, that if this term is retained, so should be a corresponding damping term in the mean flow equation (9). Nonlinear dissipation is captured by $\delta_i < 0$, which can arise (in more general formulations than the shallow-water equations) via radiation caused by quadratic self-interactions of the progressive edge waves (Mathew & Akylas 1990). This type of radiation does not occur for shallow-water edge waves since the eigenvalue of equation (5) can never intersect the continuous spectrum of equation (7*f*). The only dissipative term that does arise in the shallow-water equations is $\lambda_i < 0$. This dissipation is caused by energy leaked to radiation, which is generated at second order by quadratic interactions of the counterpropagating edge waves. In particular, it is generated by the resonance of the homogeneous solution to (7*d*) and the inhomogeneous term. The homogeneous solution always exists for a given ω^2, so that barring a zero in the eigenfunction transform of the inhomogeneous term at ω^2, this type of radiation is always present.

There are significant differences between the mean-field Davey–Stewartson equations studied by Pierce & Knobloch (1993) and the edge-wave equations, (9) and (12). The former are fully $(2+1)$-dimensional, while (9) and (12) constitute a $(1+1)$-dimensional system. Further differences are evidenced by the complex coefficients due to dissipation and radiation, the parametric forcing terms, and the additional restrictions placed on the solutions by the mean flow equations (9). In particular, in the absence of mean flows ($B = 0$) equation (9*b*) implies that $(|A^+|^2 - |A^-|^2)$ is independent of Y. However, even if this condition were initially satisfied for some spatially modulated state, it will necessarily be violated in time of $O(\epsilon^{-1})$; this is because modulations in the edge-wave envelopes, $A^{\pm}$, propagate in opposite directions at the $O(1)$ group velocity but evolve on the slower timescale $O(\epsilon^{-2})$. These states are therefore not allowed by the evolution equations unless they are supported by either a non-trivial mean flow extending into the bulk of the fluid or else an external forcing which is not spatially homogeneous. In particular, several authors have speculated on the existence of solitary edge waves (cf.

Akylas 1983), but these waves must be allowed to generate a compatible mean flow. Such a flow will in turn affect the form of the solitary wave as well as its stability properties.

We have found that a correct accounting for the mean flows raises several subtle points. The governing equation, (2), for the velocity potential, ϕ, allows a freedom in the choice of gauge up to a linear term in the variable $T = \epsilon t$. It is necessary to return to the Bernoulli equation, (1 *a*), in order to identify this term as a net shift in the surface elevation. As a result, it is necessary to specify the gauge such that the spatial average of the surface elevation is zero as $x \rightarrow \infty$. In the near-shore region, there is still a net 'set down' effect owing to the quadratic interactions of the propagating edge waves, but this corresponds to an $O(\epsilon^2)$ change in the water volume per unit length of beach and may be offset by an infinitesimal change in the water level at infinity. It is particularly important to note that this choice of gauge is not independent of the envelope amplitudes, $A^{\pm}$, and hence results in an additional mean-field term in the evolution equations. For spatially homogeneous solutions, the mean-field averaging has no effect and the mean flow appears only for travelling wave states. However, spatially dependent perturbations (i.e. modulational instabilities) are affected by the averaging and hence the modulational stability criteria do not reduce to the stability criteria for spatially homogeneous perturbations in the limit that the perturbation wavelength goes to infinity. Moreover, the effects of the mean flow do appear in the modulational stability criteria for standing waves, in spite of the fact that the unperturbed state has no mean flow.

In the face of the strong coupling between the mean flow and the edge waves one may expect the modulational dynamics of the unbounded problem to be tame unless the mean flow is externally forced. This follows from (9 *a*) which shows that the mean flow must satisfy a linear wave equation, and that mean flow disturbances propagate at the shallow-water phase speed, $(gh_0)^{\frac{1}{2}} \equiv 1$. In particular, if we consider the modulational stability of the Stokes waves, (9 *b*) shows that a modulational perturbation of the form (21 *a*), (24) generates a mean flow perturbation (21 *b*) which at leading order propagates at the group velocity, c_g. Since this perturbation also has to satisfy (9 *a*), a necessary condition for the existence of a modulational instability is that the propagation speed of mean flow disturbances matches that of the modulational disturbances, or

$$c_g = (gh_0)^{\frac{1}{2}} \equiv 1. \tag{43}$$

This condition is in fact overly strict owing to our assumption that B is independent of the slow offshore coordinate, ϵx. If B is allowed to represent obliquely propagating shallow-water waves, the equality in (43) may be replaced by $c_g \geqslant (gh_0)^{\frac{1}{2}}$. However, from (5) we have that

$$\int_0^\infty (\omega^2 a^2 - k^2 h a^2)\,\mathrm{d}x = \int_0^\infty h a'^2\,\mathrm{d}x > 0, \tag{44}$$

and hence that

$$1 > \omega/k > c_g. \tag{45}$$

The first inequality follows from the fact that ω^2 is a discrete eigenvalue of equation (5), while the second inequality follows from (44) and the expression for the group velocity, (6 *b*). Thus in the shallow-water formulation there are no modulational instabilities through the usual mechanisms. This result is an anomalous feature of the shallow-water formulation and does not necessarily hold for the full $O(1)$ depth problem since there is no strict lower bound on the phase speed of deep-water waves. If viscosity is included at leading order, one may expect that the leading-order mean flow equation, (7*a*), will yield a solution for the offshore dependence of the mean flow, $b(x)$, which decays on the fast scale. When one evaluates the solvability criterion at order [300] (from (8*a*)) all of the integrals will be bounded and independent of ϵ. Consequently, one will obtain a single inhomogeneous wave equation for the mean flow envelope, B, rather than the two separate equations (9*a*, *b*) obtained from the shallow-water equations. This type of inhomogeneous wave equation also arises in the water-wave problem on finite depth (Pierce & Knobloch 1993), and it is clear in such a formulation that modulations of the wave envelope have a causal relationship with variations in the mean flow. Similarly, it is clear that shallow-water edge waves couple to the mean flow, but since they are trapped on an $O(1)$ lengthscale they cannot generate compatible mean flows which propagate throughout the bulk. Hence the edge waves in the shallow-water formulation are relegated to a more passive role in the mean flow interaction. In addition, if there were a component of the mean flow that decayed in the offshore direction, it would contribute to the set down of the surface elevation in the near-shore region (cf. (14)). This is one possible explanation for the experimental observation that the shallow-water equations underestimate the set down associated with standing edge waves (Yeh 1986).

In addition to parametric forcing, edge waves can also be produced as solutions of the initial value problem in the unforced system ($\beta = 0$), or by sidewall forcing as in the experiments of Yeh (1985, 1986). In the latter case, travelling waves can exist as solutions and are not damped since, within the shallow-water formulation, radiation requires the existence of counterpropagating waves. For the same reason, there are no solutions in a spatially extended system in the form of counterpropagating waves, though such solutions may exist if the beach has an $O(1)$ length (Yeh 1986). As in the parametrically forced problem, modulational instabilities are not allowed in the shallow-water formulation, which apparently contradicts the results of Yeh (1985) for progressive edge waves. The experiments use a depth profile corresponding to our piecewise linear profile (cf. §4). Yeh's results indicate that the shallow-water equations describe the linear fundamental edge waves more or less accurately, though the 15° beach angle is at the outer edge of validity. Yeh observes an instability of the weakly nonlinear progressive waves which he attributes to modulational instability, though he points out that it does not follow the expected evolution of the Benjamin–Feir instability. In particular, he observes that the lower sideband grows without bound, while the upper sideband saturates at a very low level. Close inspection of the data, however, seems to indicate that the upper sideband is simply part of the smooth tail of the primary carrier frequency and decays at the same rate.

We believe that the conditions of Yeh's experiment do not conform particularly well to the restrictions implicit in our analysis. First, the dissipation is sufficiently large that the progressive waves that Yeh has studied are, strictly speaking, not homogeneous travelling wave solutions to (17) at all. Second, boundary-layer estimates of the dissipation indicate that there is a significant difference between the damping rates of the sidebands. Thus the instability could not be considered as a modulational instability governed by the evolution equations (9) and (12), since it is implicit in the derivation that the sidebands must have wavenumbers close enough to the carrier wave that they will have the same damping rates. If the lower sideband is experiencing resonant growth, this is likely to be the result of a resonant triad interaction of sidebands with an $O(1)$ wavenumber separation from the carrier wave (the sideband frequencies differ from the carrier wave by roughly 16.7%). However, we note that the dispersion relation for inviscid edge waves on an unbounded linear beach ($\omega = (gk \sin S)^{\frac{1}{2}}$, S the beach slope) does not admit solutions to the resonance equations in the neighbourhood of the carrier wave. In addition, we note that the modulations of the wavemaker could cause contamination of the low-frequency spectrum which would not necessarily be perceptible in the processed data. Such contamination can play a definite role in the modulational stability either as an inhomogeneous solution to the mean flow equations, or perhaps through a more complicated resonance mechanism. These questions may be resolved either experimentally by increasing the lengthscale of the edge waves thereby reducing the effects of viscous dissipation, or theoretically by improving the relevance of the predictions with a full solution to the $O(1)$ depth problem with viscosity and surface tension.

While we have not considered the modal interaction problem, it is interesting to speculate on the physical differences between the pure mode and mixed mode cases. In particular, progressive edge waves have often been cited as the primary cause for beach cusps, in spite of the fact that edge-wave theory has never predicted steady-state currents that vary on a spatial scale appropriate to cusp formation. This has lead several researchers to conclude that standing edge waves are the principal cause of cusp formation, although the currents generated by the standing waves are oscillatory in time and are not as efficient a mechanism for sand transport as steady-state currents. The mean flow is not a candidate for this effect since the cusps appear on a spatial scale that is presumably on the order of the fast longshore wavelength. However, quadratic interaction of linear modes with the same frequency but different wavenumbers would be responsible for generating stationary cells with longshore wavenumber that is the sum or difference of the interacting mode wavenumbers (for counterpropagating and copropagating modes, respectively) and independent of the fast timescales. This is similar to the [220] term in our expansion (3), but there would be no reason to expect that the driving inhomogeneous term would vanish identically as it does in (7*e*). These stationary cells give precisely the steady-state flow patterns one would expect to generate cusps, so that the existence of pronounced beach cusps might be expected to coincide with the existence of mixed modes. It is important to note that these cells will be formed in response to interactions of both counterpropagating and copropagating modes. In particular, cusp formation by this mechanism could occur for travelling wave states as well as standing waves. Guza & Chapman (1979) point out that cusp formation under experimental conditions with non-reflective boundaries appears to depend on almost exact normal incidence of the driving waves. Since the experiments are designed to excite a single linear mode, this would indicate that the primary experimental mechanism for cusp formation is indeed standing waves. However, they also conclude that cusps on a natural beach should be surprising since the exact normal incidence or perfectly reflective endwalls which favour standing waves are not common, in contradiction with numerous field observations. In fact, natural systems are typically driven with a large range of frequencies and incident directions simultaneously and cannot adhere to the single linear mode restriction as well as experiments do. Consequently, mode interactions and steady-state currents of the type discussed above are not unlikely. Guza & Chapman's observations thus lend credence to the hypothesis that mixed edge-wave modes may be an important mechanism for the formation of beach topography and rip currents.

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